For the solution of the outside flow, Eqs. (8) and (9) can be combined into one single integral equation of the form

$$ {}_{iv}v_0{}^2 - \left(\frac{2\gamma}{\gamma+1}\right) {}_{iv}v_0 + \frac{c_2}{c_1{}^2}\left[\frac{2(\gamma-1)}{(\gamma+1)}\right] = \left[\frac{4m(\gamma-1)\sigma c_1{}^6}{\Gamma(\gamma+1)nR^4}\right]\int_{-\infty}^{+\infty}\left[{}_{iv}v_0(\xi') - {}_{iv}v_0{}^2(\xi')\right]^4 \operatorname{sgn}(\xi-\xi') \times e^{-|\xi-\xi'|}\,d\xi' \quad (12) $$

The subscript *iv* now stands for the distribution outside the shock. This equation had been subjected to thorough investigations in Refs. 1 and 2. Its solution supplies the boundary conditions at $\xi = 0^-$ and $\xi = 0^+$ and can be used to evaluate the definite integral J in the shock-structure problem. With this in mind, the shock-structure solution of Eqs. (7) can be expressed in either one of the two forms

$$ y - y_0{}^- = \int_{{}_{iv}v_0(0^-)}^{v_0} \frac{v_0 dv_0}{(\gamma+1)/2\gamma\, v_0{}^2 - v_0 + (\gamma-1)/2\gamma\,(C - QJ)} \quad (13) $$

$$ y\left(\frac{\gamma+1}{2\gamma}\right) = \left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{L\Gamma}{\mu}\right) \times $$

$$ = \left[\frac{{}_{iv}v_0(0^-)}{{}_{iv}v_0(0^-) - {}_{iv}v_0(0^+)}\right]\ln[{}_{iv}v_0(0^-) - V_0] - \left[\frac{{}_{iv}v_0(0^+)}{{}_{iv}v_0(0^-) - {}_{iv}v_0(0^+)}\right]\ln[v_0 - {}_{iv}v_0(0^+)] \quad (14) $$

Equation (14) bears essentially the same form as the classical Becker's solution for Prandtl number = 1 and μ = const.[7] However, the shape of the distribution will be different because ${}_{iv}v_0(0^+)$ and ${}_{iv}v_0(0^-)$ are different from the corresponding values across a radiationless shock, with the same upstream conditions. From Eq. (12), we observe that the condition of having a discontinuity in ${}_{iv}v_0$ across the shock is to require Eq. (18) to have real roots at $\xi = 0^-$ and 0^+. This condition is

$$ \left[\frac{\gamma^2}{2(\gamma^2-1)}\right] \geq \left[C - J\,\frac{2n\sigma c_1{}^6}{3\Gamma c_p{}^4}\right] $$

where J is defined in Eq. (11). Since J is never positive for a real shock, an important effect of thermal radiation is to reduce the velocity jump across the shock, and to increase the upstream Mach number necessary to produce a shock. These findings are in agreement with those of Heaslett and Baldwin[2] and show how the behavior across the discontinuity establishes the proper boundary conditions for their analysis. Results of Eq. (14) are plotted for several cases in Fig. 1.

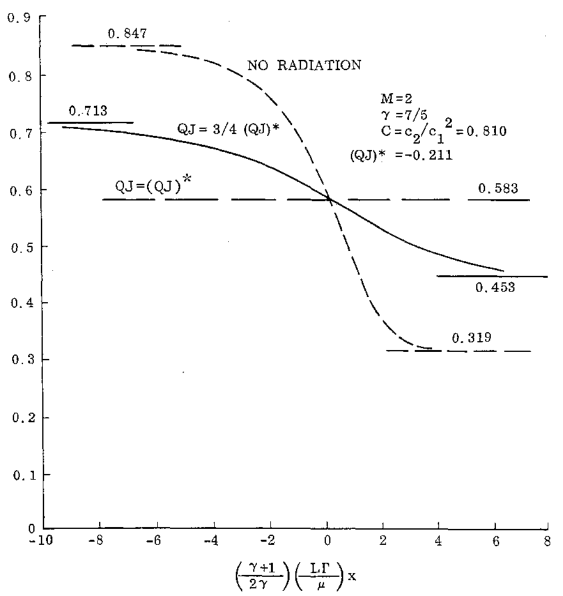

Fig. 1 Radiation effect on first-order structure of a typical shock.

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Linear Jet and Wake Solutions with Pressure Gradients

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Recently, Steiger and Bloom[1] presented the linear similar solutions for laminar jet and wake flows with pressure gradients. In the present paper the motivation is to establish a parallel and concise record of the similar solutions for turbulent flows. These solutions represent a useful and simple entré into the understanding of the more general flows with arbitrary pressure gradients.

In order to provide a convenient comparison, it was considered worthwhile to review the laminar flows and, in the process, some new considerations have been added to the work of Ref. 1. For our present purpose, we wish simplicity and so the development will be restricted to incompressible flow. The basic equations are

$$ (\partial/\partial x)(y^r u) + (\partial/\partial y)(y^r v) = 0 \quad (1) $$

$$ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = UU' + \frac{\nu_e}{y^r}\frac{\partial}{\partial y}\left(y^r\frac{\partial y}{\partial y}\right) \quad (2) $$

where r = 0 or 1 for either plane flow or axisymmetric flow, ν_e is the "effective" viscosity and is either the molecule viscosity in the case of laminar flow or the eddy viscosity in the case of turbulent flow. The boundary conditions are that $v(0) = \partial u(0)/\partial y = 0$ and $u \to U$ exponentially as $y \to \infty$. We now seek similar solutions of the form

$$ u(x,y) = U(x) + u_c(x)f(\eta) \qquad \eta = y/b(x) \quad (3) $$

where U is the mainstream velocity and $u_c(x) = u(x,0) - U(x)$ is the centerline difference velocity. If (3) is inserte

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into (2) and terms are neglected according to the condition $u_c/U \ll 1$, we obtain

$$f'' + \frac{r}{\eta^r} f' + \left[\frac{U'bb}{\nu_e} + \frac{U'b^2}{\nu_e(r+1)}\right]\eta f' - \left[\frac{b^2 U}{\nu_e}\frac{u_c'}{u_c} + \frac{b^2 U'}{\nu_e}\right] f = 0 \quad (4)$$

Since $f = f(\eta)$, the quantities in the square brackets must be constant. We therefore set

$$(Ub'b/\nu_e) + [U'b^2/\nu_e(r+1)] = 2 \quad (5)$$

$$(Ub^2/\nu_e)(u_c'/u_c) + (b^2U'/\nu_e) = -4\lambda \quad (6)$$

resulting in

$$f'' + [(r/\eta^r) + 2\eta]f' + 4\lambda f = 0 \quad (7)$$

The values 2 and -4λ are chosen for later convenience. In effect they define the length scale b.

As discussed by Steiger and Bloom[1] and in more detail by Mellor,[3] there is a discrete, infinite set of eigenfunctions, which satisfy the boundary conditions $f(0) = 1$ and $f \to 0$ exponentially as $\eta \to \infty$. The complete set would allow us to satisfy any initial profile at $x = 0$. However, all solutions but one contribute zero momentum flux, and that solution decays more slowly than the rest. The nonzero momentum-flux solution corresponds to $\lambda = (r+1)/2$ or $\lambda = \frac{1}{2}$ for the plane case and to 1 for the axisymmetric case. This result is also obtained by integrating (7) from $\eta = 0$ to ∞.

Thus, if $\lambda = (r+1)/2$, the solution is

$$f = e^{-\eta^2} \quad (8)$$

for all of the cases. Since

$$\int_0^\infty e^{-\eta^2} d\eta = \frac{\pi^{1/2}}{2}$$

it is probably useful, in the case of an exponentially measured profile, to define b according to

$$b = \left(\frac{2}{\pi^{1/2}}\right)\int_0^\infty \left[\frac{(U-u)}{u_c}\right] dy$$

Laminar Flow

We stipulate that $\nu_e = \nu$. If we independently set $U \sim x^m$, then solutions to Eqs. 5 and 6 are like $b^2 \sim x^{1-m}$ and $u_c \sim x^n$ where $n = -(1+m)/2,\ -1$ for $r = 0, 1$. This is the conventional form of presenting the streamwise variation of the flow quantities. Here m is the independent parameter governing the flow.

A more complete representation is to incorporate a set of initial conditions so that $U = U_0$, $b = b_0$, and $u_c = u_{c0}$ at $x = 0$. Solutions of Eqs. (5) and (6) are

$$U/U_0 = [1 + \Lambda\tilde{x}/m]^m \quad (9a)$$

$$b/b_0 = [1 + \Lambda\tilde{x}/m]^{(1-m)/2} \quad (9b)$$

$$u_c/u_{c0} = [1 + \Lambda\tilde{x}/m]^n \quad (9c)$$

where

$$\tilde{x} \equiv (\nu/U_0b_0)(x/b_0) \quad (10a)$$

$$\Lambda \equiv U'b^2/\nu \quad (10b)$$

$$m \equiv \begin{cases}\Lambda/(4-\Lambda) & r = 0\\ \Lambda/4 & r = 1\end{cases} \quad (10c)$$

$$n \equiv \begin{cases}-(2+\Lambda)/(4-\Lambda) & r = 0\\ -(4+\Lambda)/4 & r = 1\end{cases} \quad (10d)$$

Here $\Lambda = U'b^2/\nu$ emerges as the principal independent parameter and may be set experimentally, purely from local conditions. Equations (9a–9c) may be related easily to experimental observables. Within the range $-\infty < \Lambda < \infty$, there are some particularly simple or interesting cases.

1) When $\Lambda = 0$, we find, of course, that $U = U_0$; also $b/b_0 = [1 + 4\tilde{x}]^{1/2}$. For the plane case, $u_c/u_{c0} = [1 + 4\tilde{x}]^{-1/2}$, whereas in the axisymmetric case, $u_c/u_{c0} = [1 + 4\tilde{x}]^{-1}$.

2) In the plane flow case we find that, when $\Lambda = 4$, $U = U_0 \exp(4\tilde{x})$, $b = b_0 \exp(-2\tilde{x})$, and $u_c = u_{c0}\exp(-2\tilde{x})$. This result is the limiting condition as $m \to \pm\infty$. It points out one of the benefits derived from the use of the full solution, satisfying initial conditions and the condition $\Lambda =$ const. Equation (8) is continuous as Λ passes through the value 4, whereas the equation $U \sim x^n$, for example, is not.

3) An interesting case is where $u_c/U =$ const or where, relatively, the wake does not decay at all. For plane flow this occurs when $\Lambda = -1$ and for axisymmetric flow when $\Lambda = -2$.

It is of interest to relate the preceding linear solutions to the exact solutions for incompressible plane wakes recently obtained by Kennedy.[2] Kennedy included the nonlinear terms in the differential equation, in which case similar solutions only exist for $u_c/U =$ const.

Roughly, we find that, as long as $0.5 < u_c/U \leq 0$, Kennedy's solution may be approximated by the linear solution where $\Lambda = -1$.† Of course, the linear and the exact solutions are coincident when $u_c/U \to 0$.

Turbulent Flow

To carry our analysis forward for turbulent flows, it is necessary to interject a hypothesis concerning the eddy viscosity. Part of the hypothesis has already been made, i.e., that ν_e is a function of x only. The complete hypothesis will be that

$$\nu_e = ku_cb \quad (11)$$

where k is a constant. Equation (11) has been verified experimentally for the constant pressure case where, for plane flow, $k = 0.078$ (Ref. 4) and for the axisymmetric case where $k = 0.67$ (Ref. 5). There is, of course, no a priori assurance that k will be the same constant or even that the form of Eq. (11) is valid for nonzero pressure gradients.

Presuming the validity of the hypothesis, solutions of Eqs. (5) and (6) are

$$U/U_0 = [1 + \Lambda\tilde{x}/m]^n \quad (12a)$$

$$b/b_0 = [1 + \Lambda\tilde{x}/m]^q \quad (12b)$$

$$u_c/u_{c0} = [1 + \Lambda\tilde{x}/m]^n \quad (12c)$$

where

$$\tilde{x} \equiv u_{c0}x/U_0b_0 \quad (13a)$$

$$\Lambda \equiv U'b/u_c \quad (13b)$$

$$m = \begin{cases}\Lambda/(\Lambda + 4k) & r = 0\\ 2\Lambda/(3\Lambda + 12k) & r = 1\end{cases} \quad (13c)$$

$$q = \begin{cases}-(\Lambda - 2k)/(\Lambda + 4k) & r = 0\\ -(\Lambda - 4k)/(3\Lambda + 12k) & r = 1\end{cases} \quad (13d)$$

$$n = \begin{cases}-(\Lambda + 2k)/(\Lambda + 4k) & r = 0\\ -\frac{2}{3} & r = 1\end{cases} \quad (13e)$$

Again, in the range $-\infty < \Lambda < \infty$, there are some particularly simple or interesting cases to be noted.

† Conversely, consider the exact solution where $u_c/U =$ const $= 0.463$ ($\beta = -0.40$). Using the definition

$$b = \left(\frac{2}{\pi^{1/2}}\right)\int_0^\infty \left[\frac{(U-u)}{u_c}\right] dy$$

we find that this gives $\Lambda = -1.192$ and that $u/U = (u_{c0}/U_0)[1 + 5.2\tilde{x}]^{0.075}$. The term in square brackets is the error.

1) When $\Lambda = 0$ we find, of course, that $U = U_0$; also $b/b_0 = [1 + 4k\tilde{x}]^{1/2}$ and $u_c/u_{c0} = [1 + 4k\tilde{x}]^{-1/2}$ for the plane case; and $b/b_0 = [1 + 6k\tilde{x}]^{1/3}$ and $u_c/u_{c0} = [1 + 6k\tilde{x}]^{-2/3}$ for the axisymmetric case.

2) As in the laminar flows, there exist exponential solutions but in the axisymmetric case as well as in the plane case. In the plane case, we find that when $\Lambda = -4k$, then $U/U_0 = \exp(-4k\tilde{x})$, $b/b_0 = \exp(6k\tilde{x})$, and $u_c/u_{c0} = \exp(2k\tilde{x})$. For the axisymmetric case we also find that when $\Lambda = -4k$, then $U/U_0 = \exp(-4k\tilde{x})$, but $b/b_0 = \exp(4k\tilde{x})$ and $u_c/u_{c0} = 1$.

3) The case where u_c/U = const occurs when $\Lambda = -k$ in the plane case and $\Lambda = -2k$ in the axisymmetric case.

Hill[6] has presented some experiments and theory on turbulent plane wakes in pressure gradients. One of his experiments corresponds rather closely to the condition u_c/U = const = 0.5, and we find that $\Lambda \simeq -k = -0.078$. Thus, as Hill comments, the value k seems to be independent of the pressure gradient; this is analogous to Clauser's discovery[7] in the case of "equilibrium," turbulent boundary layers.

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An Energy Inventory in a Coaxial Plasma Accelerator Driven by a Pulse Line Energy Source

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Introduction

THE most severe limitation of the many plasma accelerators that have been considered for space-propulsion applications is low electrical efficiency.[1] In order to improve the efficiency of a specific accelerator it is necessary to know how the source energy is partitioned within the accelerator at any time.

In the experiments described in this paper a pulsed coaxial plasma gun, powered by a pulse-line energy source, was studied. The partition of energy among the source, the magnetic field, and the work done on the plasma was determined from measurements of the magnetic field within the gun, together with the voltage and current at the terminals of the gun. A pulse line energy source[2] was used because it can deliver a constant current at a constant voltage for most of its period.

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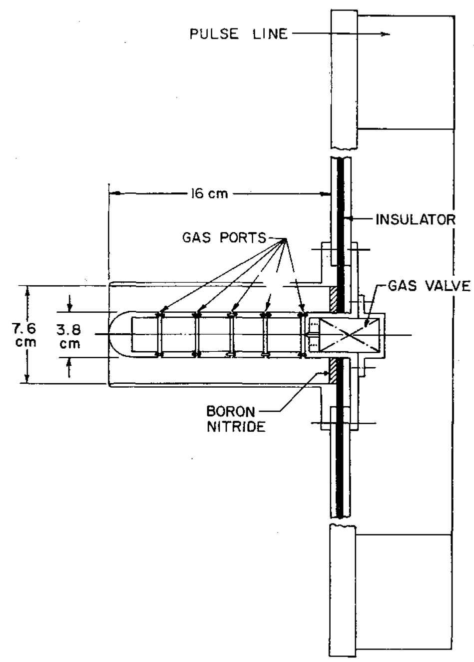


Fig. 1 Schematic diagram of the accelerator.

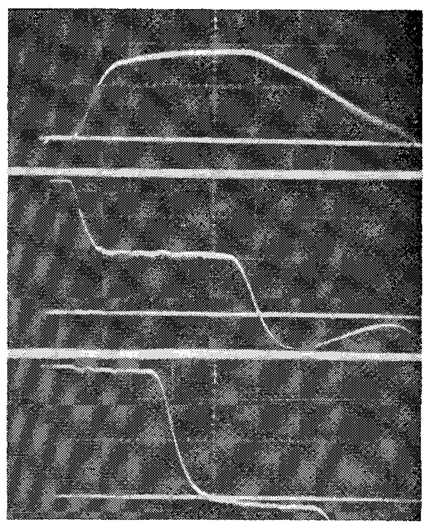
Description of the Apparatus

The plasma accelerator[2] is illustrated schematically in Fig. 1. The energy storage capacitor is a single unit, wound in the form of a toroid with an outside diameter of 22 in., an inside diameter of 19 in., and 12 in. in length. This capacitor exhibits a pulse-line behavior because the wave propagation time in the capacitor is comparable to the period of the system.[2] The line impedance is approximately 17 $m\Omega$, the pulse time 0.8 μsec, and the total capacitance 22 μF. The unit is operated at 6.3 kv with nitrogen as propellant. The propellant is injected through a solenoidal gas valve and is dispensed through multiple gas ports into the

Current

Voltage on the gun

Voltage at the back of the pulse line



Time→

Fig. 2 Current and voltage waveforms for the pulse line: $I \sim 10^5$ amp/cm, $V \sim 2$kv/cm, $t = 0.2\ \mu$ sec/cm.